THREE ALGEBRAIC PROOFS IN SET THEORY — TWO OF WHICH ARE PROOFS OF THE SAME STATEMENT

EXAMPLE 1:

TO PROVE: FOR ALL SETS A, B, and C,

$$(A \cup B) - C = (A - C) \cup (B - C).$$

PROOF: Let A, B and C be any sets.

$\therefore (A \cup B) - C = (A \cup B) \cap C^c$
by the SET DIFFERENCE LAW,

$= C^c \cap (A \cup B)$, by the Commutative LAWS,

$= (C^c \cap A) \cup (C^c \cap B)$
by the DISTRIBUTIVE LAWS,

$= (A \cap C^c) \cup (B \cap C^c)$ by the Commutative Laws (applied twice),

$= (A - C) \cup (B - C)$
by the Set DIFFERENCE LAWS (APPLIED TWICE).

$\therefore (A \cup B) - C = (A - C) \cup (B - C)$ by the transitive property of "=".

$\therefore$ FOR ALL SETS A, B, and C,

$$(A \cup B) - C = (A - C) \cup (B - C),$$

by DIRECT PROOF. Q.E.D.

EXAMPLE 2:

TO PROVE: FOR ALL SETS A and B,

$$A \cap ((B \cup A^c) \cap B^c) = \emptyset.$$

PROOF: Let A and B be any sets.

$A \cap ((B \cup A^c) \cap B^c) = A \cap (B^c \cap (B \cup A^c))$

by the Commutative Laws,

$= (A \cap B^c) \cap (B \cup A^c)$

by the Associative LAWS,

$= ((A \cap B^c) \cap B) \cup ((A \cap B^c) \cap A^c)$

by the Distributive Laws,

$= (A \cap (B^c \cap B)) \cup ((B^c \cap A) \cap A^c)$

by the Associative Laws AND the Commutative LAWS,

$= (A \cap (B \cap B^c)) \cup (B^c \cap (A \cap A^c))$

by the Commutative Laws AND the Associative Laws,

$= (A \cap \emptyset) \cup (B^c \cap \emptyset)$

by the Complement Laws (applied twice),

$= \emptyset \cup \emptyset$ by the UNIVERSAL BOUND LAWS (applied twice),

$= \emptyset$ by the IDENTITY LAWS.

$\therefore$ By transitivity, $A \cap ((B \cup A^c) \cap B^c) = \emptyset$.

$\therefore$ FOR ALL SETS A and B,

$A \cap ((B \cup A^c) \cap B^c) = \emptyset$, by DIRECT PROOF.

QED

EXAMPLE 3:

TO PROVE: FOR ALL SETS A AND B,

$$A \cap ((B \cup A^c) \cap B^c) = \emptyset.$$

PROOF: LET A and B be any sets.

$$A \cap ((B \cup A^c) \cap B^c) = A \cap (B^c \cap (B \cup A^c))$$

by the Commutative Laws,

$$= (A \cap B^c) \cap (B \cup A^c)$$

by the Associative Laws,

$$= (A \cap B^c) \cap [(B \cup A^c)^c]^c$$

by the Double Complement Law,

$$= (A \cap B^c) \cap [B^c \cap (A^c)^c]^c$$

by DeMORGAN'S LAWS,

$$= (A \cap B^c) \cap (B^c \cap A)^c$$

by the Double Complement Law,

$$= (A \cap B^c) \cap (A \cap B^c)^c$$

by the Commutative Laws,

$$= \emptyset$$ by the Complement Laws.

$\therefore$ $A \cap ((B \cup A^c) \cap B^c) = \emptyset$, by Transitivity of "=".

$\therefore$ FOR ALL SETS A and B,

$A \cap ((B \cup A^c) \cap B^c) = \emptyset$, by DIRECT PROOF.

Q.E.D.